Mathematical Analysis - List 6

1. Use the Heine definition of a limit to show that:

a) $\lim\limits_{x\to\frac{\pi}{2}^+} \operatorname{sign}(\cos x) = -1$; b) $\lim\limits_{x\to -3^-} \sqrt{x^2-9} = 0$;

c) $\lim\limits_{x\to\infty} \dfrac{1-2x^3}{x^3+1} = -2$; d) $\lim\limits_{x\to\ 2^+} \dfrac{1}{x-2} = \infty$; e) $\lim\limits_{x\to\ 1} \dfrac{x-3}{|x^2+2x-3|} = -\infty$.

2. Show that the following limits do not exist:

a) $\lim\limits_{x\to 3} \dfrac{x^2}{x-3}$; b) $\lim\limits_{x\to 2} \dfrac{x}{4-x^2}$; c) $\lim\limits_{x\to\infty} \sin\sqrt{x}$;

d) $\lim\limits_{x\to 0} \dfrac{\operatorname{sign} x}{\operatorname{sign}(x+1)}$; e) $\lim\limits_{x\to\pi} \dfrac{1}{\sin x}$; f) $\lim\limits_{x\to 0^-} \cos\dfrac{1}{x^2}$.

3. Find the left- and right-hand limits and decide if the limit exists:

a) $\lim\limits_{x\to 0} x \operatorname{sign} \mathrm{x}$; b) $\lim\limits_{x\to 2} \dfrac{x^2-4}{|x-2|}$;

c) $\lim\limits_{x\to 1} \dfrac{|x-1|^3}{x^3-x^2}$; d) $\lim\limits_{x\to 0} \dfrac{\sin x}{|x|}$.

4. Use the Limit Laws given in class to evaluate the limits:

a) $\lim\limits_{x\to 1} \dfrac{x^3-1}{x^4-1}$; b) $\lim\limits_{x\to 64} \dfrac{\sqrt[3]{x}-4}{\sqrt{x}-8}$; c) $\lim\limits_{x\to\infty} \dfrac{x^2-5x+4}{x(x-5)}$;

d) $\lim\limits_{x\to 1} \dfrac{x^6-1}{1-x^2}$; e) $\lim\limits_{x\to 6} \dfrac{\sqrt{x-2}-2}{x-6}$; f) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{2x}$.

5. Use the fact that $\lim\limits_{\alpha\to 0} \dfrac{\sin\alpha}{\alpha} = 1$ to evaluate the following limits:

a) $\lim\limits_{x\to 0} \dfrac{\sin^2 3x}{x^2}$; b) $\lim\limits_{x\to 0} \dfrac{\sin\frac{x}{2}}{\sin\frac{x}{3}}$; c) $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\cos 5x}{\cos 3x}$; d) $\lim\limits_{x\to 0} \dfrac{\sin x^3 \sin x^7}{\sin x^4 \sin x^6}$;

e) $\lim\limits_{x\to\infty} \dfrac{\operatorname{tg}\frac{1}{x}}{\operatorname{tg}\frac{2}{x}}$; f) $\lim\limits_{x\to 0^-} \dfrac{\operatorname{tg} 3x}{x^3}$; g) $\lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{\operatorname{tg} x}{\operatorname{tg} 5x}$; h) $\lim\limits_{x\to 0} \dfrac{\cos 3x - \cos 7x}{x^2}$.

6. If $3x \leqslant f(x) \leqslant x^3+2$ for $0 \leqslant x \leqslant 2$, evaluate $\lim\limits_{x\to 1} f(x)$.

7. Sketch the graph of an example of a function that satisfies all of the given conditions:

a) $\lim\limits_{x\to-\infty} u(x) = \infty$, $\lim\limits_{x\to 0^-} u(x) = 1$, $u(2) = 0$, $\lim\limits_{x\to\infty} u(x) = -1$;

b) $\lim\limits_{x\to\infty} v(x) = e$, $\lim\limits_{x\to 2} v(x) = 0$, and v is an even function;

c) $\lim\limits_{x\to-\infty} f(x) = 0$, $\lim\limits_{x\to 1} f(x) = 3$, $\lim\limits_{x\to\infty} f(x) = -\infty$;

d) $\lim\limits_{x\to-\infty} g(x) = \infty$, $\lim\limits_{x\to 0^-} g(x) = -\infty$, $\lim\limits_{x\to 0^+} g(x) = 1$, $\lim\limits_{x\to\infty} g(x) = 5$;

e) $\lim\limits_{x\to-\infty} h(x) = -4$, $\lim\limits_{x\to -1} h(x) = \infty$, $\lim\limits_{x\to\infty} h(x) = 4$;

f) $\lim\limits_{x\to 1} p(x) = \infty$, $\lim\limits_{x\to 2} p(x) = 0$, and p is a periodic function with $T = 3$;